Lecture # 9

More Examples From Chapter 3

Recall from Lecture 7 that

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^c)P(E^c)}$$

More generally if $\{F_i\}_{i=1}^n$ is a partition of the sample space, then

$$P(F_j|E) = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^n P(E|F_i)P(F_i)}.$$

This is called "Bayes' Theorem."

Ex: A pile of playing cards consists of

- 4 aces
- 2 kings
- 2 queens.

A second pile consists of:

- 1 Ace
- 4 Kings
- 3 Queens.

- you choose a card from the first pile and shuffle it into the second pile
- you then draw a card from the second pile.

<u>Q:</u> If the card drawn from the second pile is an ace, what is the probability that the first card was an ace as well?

Let A = ace was drawn.

K = King was drawn.

Q = queen " "

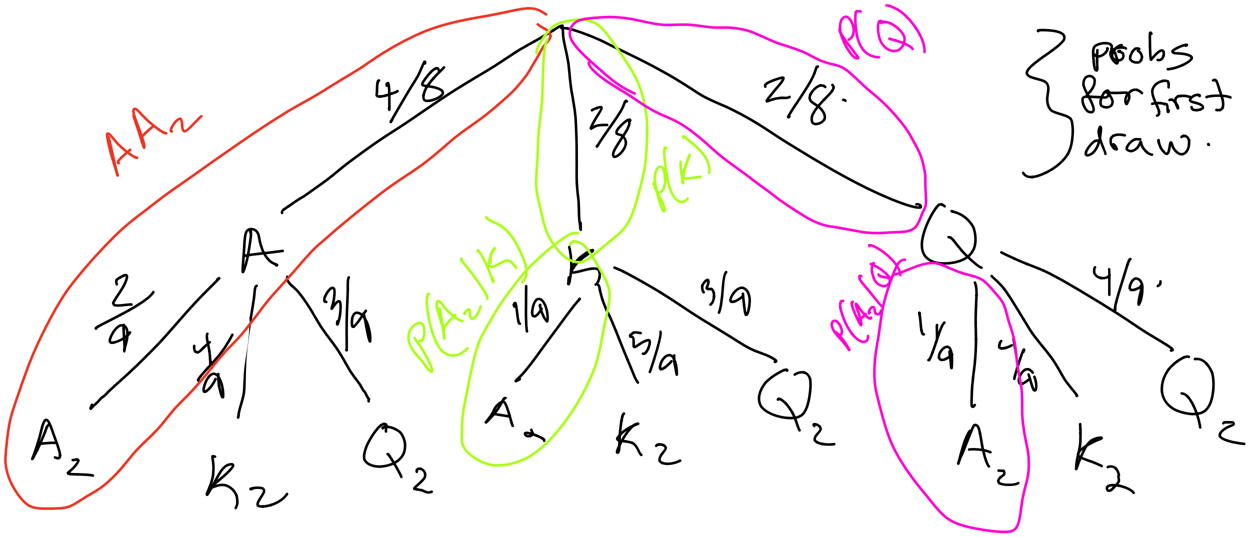
Let A_2, K_2, Q_2 be the event that a $\underline{A, K, Q}$ was drawn from the 2^{nd} pile..

So, we want $P(A \mid A_2) = \dfrac{P(AA_2)}{P(A_2)}$.

By Bayes theorem

$$P(A \mid A_2) = \frac{P(A|A_2)\,P(A)}{P(A_2|A)P(A) + P(A_2|K)P(K) + P(A_2|Q)P(Q)}$$

we can draw a diagram.



So $P(A|A_2) = \dfrac{(4/8)(2/9)}{(2/9)(4/8) + (2/8)(1/9) + (1/9)(2/8)} = 2/3.$

Ex: Suppose 10% of students attend football games and 15% of first year students attend football games. Choose a student at random.
Let A = the student attends football games.
FY = the student is a first year.

$\frac{15}{100} = \frac{3}{20}$

Are A and FY Independent?

Soln: we have $P(A) = 1/10$, $P(A|FY) = \frac{3}{20}$
We don't know $P(FY)$, but we know that $P(FY) \neq 0$. Since $P(A|FY) \neq P(A)$ and $P(FY) \neq 0$, they are not independent.

Ex: - We roll a pair of dice.
- Each roll is independent of the last.

Q: What is the probability that we roll a 5 before we roll a 7.?

A: Let E = "a 5 occurs before a 7".

Let F = the first roll is a 5.

G = the first roll is a 7.

H = the first roll is neither 5 nor 7.

By the law of total prob. and the multiplication rules,

$$P(E) = P(E|F)P(F) + P(E|G)P(G) + P(E|H)P(H)$$

Now $P(E|F) = 1$.

$P(E|G) = 0$.

$P(E|H) = P(E)$ since the roll's are indep.

$P(F) = 4/36$ $\big((1,4),(2,3),(3,2),(4,1)\big)$.

$P(G) = 6/36$ $\big((1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\big)$

$P(H) = P((F \cup G)^c) = 1 - P(F \cup G) = 1 - \left(\frac{4}{36} + \frac{6}{36}\right) = 1 - \frac{10}{36} = \frac{26}{36}.$

So $P(E) = 1\cdot\left(\frac{4}{36}\right) + 0\cdot\left(\frac{6}{36}\right) + P(E)\left(\frac{26}{36}\right)$

$$P(E) - P(E)\left(\frac{26}{36}\right) = \frac{4}{36}$$

$$P(E)\left(1 - \frac{26}{36}\right) = \frac{4}{36} \Rightarrow P(E) = \frac{4}{\cancel{36}}\cdot\frac{\cancel{36}}{10} = \frac{4}{10} = 2/5.$$

<u>Example</u> (The Monty Hall Problem)

You are on a game show and the host presents you with three closed doors. Behind two doors are goats and behind the third is a car. You choose a door. Before the door is opened, the host opens one of the remaining two doors to reveal a goat (the host knows, and always chooses the/a goat door). You now have the option to open your door or switch to the other closed door. What do you do?

Suppose the doors are labelled 1, 2, 3. Without loss of generality suppose you have chosen door #1 and that the host opens door #3.

Let E_i = the prize is behind door $\#i$. $P(E_i) = 1/3$.

Let F_i = the host opens door $\#i$.

$P(F_3 | E_1) = 1/2$, since you have chosen door # 1 and the two remaining goat doors are chosen with equal prob.

$P(F_3 | E_2) = 1$: the host can only pick door 3, since the do not want to reveal the prize.

$P(F_3 | E_3) = 0$, since they do not want to reveal the prize.

<u>So now</u>: we have door # 1, the host opens door # 3. to reveal a goat. we can either keep door # 1 or switch to # 2.

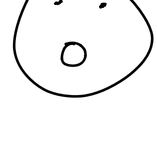
$$P(E_2 | F_3) = \frac{P(F_3 | E_2)\, P(E_2)}{P(F_3 | E_1) P(E_1) + P(F_3 | E_2) P(E_2) + P(F_3 | E_3) P(E_3)}$$

$$= \frac{1 \cdot (1/3)}{(1/2)(1/3) + 1 (1/3) + 0 \cdot (1/3)} = \underline{2/3.}$$

<u>But:</u>

$$P(E_1 | F_3) = \frac{P(F_3 | E_1) P(E_1)}{P(F_3 | E_1) P(E_1) + P(F_3 | E_2) P(E_2) + P(F_3 | E_3) P(E_3)}$$

$$= \frac{(1/2)(1/3)}{(1/2)(1/3) + 1 \cdot (1/3) + 0 \cdot (1/3)} = 1/3.$$



So you should switch!!